Spin Squeezing under Non-Markovian Channels by Hierarchy Equation Method

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We study spin squeezing under non-Markovian channels, and consider an ensemble of N independent spin-1/2 particles with exchange symmetry. Each spin interacts with its own bath, and the baths are independent and identical. For this kind of open system, the spin squeezing under decoherence can be investigated from the dynamics of the local expectations, and the multi-qubit dynamics can be reduced into the two-qubit one. The reduced dynamics is obtained by the hierarchy equation method, which is a exact without rotating-wave and Born-Markov approximation. The numerical results show that the spin squeezing displays multiple sudden vanishing and revival with lower bath temperature, and it can also vanish asymptotically.

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I. INTRODUCTION

Spin squeezing has attracted much attention for decades [1–8]. An important application of spin squeezing is to detect quantum entanglement [9–11]. As a multipartite entanglement witness, spin squeezing is relatively easy to be generated and measured [2, 12–14]. Many efforts have been devoted to find relations between spin squeezing and entanglement [1–7, 15–17]. Another application of spin squeezing is to improve the precision of measurements. For example, spin squeezing plays an important role in making more precise atomic clock [2, 6, 18, 19] and gravitational-wave interferometers [20–22], and so on.

Spin-squeezed states are useful resources for quantum information processing. However, in practice, decoherence is inevitable and harmful to spin squeezing and entanglement [23–29]. Generally, when the system-bath coupling strength is weak enough, the decoherence is studied by using the master equation method, which is derived by employing the Born approximation [23, 24]. Besides, the Markov approximation can be applied if the time scale of the bath is much shorter than that of the system. To overcome the above approximations, a set of hierarchical equations were established by Tanimura *et al* [30–36]. It provides an exact way to obtain the reduced dynamics of system [37, 38]. However, for numerical reasons, it is hard to treat systems with large number of particles straightforwardly. Here, we show that for the open system we consider, we can reduce the multiparticle dynamics to the two-particle one, and then we efficiently use the hierarchy equation method to make numerical calculations.

As we know, spin squeezing is a multipartite entanglement witness. Reference [39] has shown that for a many-particle system with exchange symmetry, the spin squeezing parameters of the total system can be expressed in terms of local expectations and correlations. Here, we consider such an ensemble of N independent spin-1/2 particles. Each particle interacts with its own bath, and the baths are independent and identical. Thus, the exchange symmetry is not affected by the decoherence, and the spin squeezing parameters of the open system can be expressed by dynamics of the local expectations and correlations. For the system under consideration, we find that the dynamics of any two particles is governed only by the local Hamiltonian of the two particles and their baths. Then, we use the hierarchy equation method to calculate the dynamics the the local expectations and correlations. Reference [39] has also shown that the spin squeezing has close relation with pairwise entanglement if the state of the collective spin system lies in the $J = N/2$ sector, where J is the collective angular momentum of the system. Therefore, since the state of the system will not lie in $J = N/2$ sector anymore under decoherence, the ability of spin squeezing in detecting pairwise entanglement needs to be further studied and clarified.

This paper is organized as follows. In Sec. II, we introduce the Hamiltonian and the initial state of the open system. The definition of the spin squeezing parameters is given in Sec. III, and we also discuss the symmetry of the system and reduce the multi-qubit dynamics into the two-qubit one. In Sec. IV, we introduce the hierarchy method and give a alternative form of the hierarchy equation. We numerically calculate spin squeezing parameters and the rescaled concurrence of the open system under decoherence and compare their behaviors in Sec. IV. At last, a summary is given in Sec. V.

II. HAMILTONIAN AND INITIAL STATE

The system we consider is an ensemble of N independent spin-1/2 particles with exchange symmetry, and each particle interacts with its own bosonic bath. The N baths are independent and identical. The Hamiltonian

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of the total system is ($\hbar = 1$)

$$\begin{aligned} H &= H_S + H_B + H_{SB} \\ &= \sum_{\alpha=1}^{N} \frac{\omega_0}{2}\sigma_{\alpha z} + \sum_k \omega_k a_k^\dagger a_k + \\ & \sum_{\alpha=1}^{N} \sum_k g_{\alpha k}\sigma_{\alpha x}\left(a_k^\dagger + a_k\right), \end{aligned} \tag{1}$$

where the first term is the Hamiltonian of the system with $\sigma_{k\alpha}(\alpha = x, y, z)$ the Pauli matrices for the k-th spin and ω_0 the frequency for all qubits. The second term describes the bosonic bath, where b_k and $b_k^\dagger$ are the creation and annihilation operators of the k-th mode with frequency ω_k. The system-bath coupling is characterized by the third term with $g_{\alpha k}$ the coupling strength for qubit α. Here, we study N independent baths, i.e., the bath can be divided into N parts and $g_{\alpha k}$ is only non-zero when mode k belongs to the α-th part.

The initial state of the total system is set to be a product state

$$\rho_T(0) = \rho_S(0) \otimes \rho_B(0), \tag{2}$$

where $\rho_S(0)$ is a spin-squeezed state and $\rho_B(0)$ is a thermal state given by

$$\rho_B(0) = \prod_k \frac{\exp(-\beta\omega_k a_k^\dagger a_k)}{Z_k} \tag{3}$$

with the inverse temperature $\beta = 1/(k_B T)$ and partition function $Z_k = \mathrm{Tr}\exp(-\beta\omega_k a_k^\dagger a_k)$ for mode k. In this paper we take $k_B = 1$.

We choose the initial state as a standard one-axis twisted state [1]

$$|\Psi(0)\rangle = e^{-i\theta J_x^2/2}|\downarrow \ldots \downarrow\rangle \tag{4}$$

with

$$J_\alpha = \frac{1}{2}\sum_{k=1}^{N} \sigma_{k\alpha} \tag{5}$$

the total angular momentum operators. This state is prepared by the one-axis twisted Hamiltonian $H = \chi J_x^2$, with the coupling constant χ , and $\theta = 2\chi t$ the twist angle. For our case, the system of N spin-1/2 behaves like an effective large spin $N/2$.

III. SPIN SQUEEZING AND REDUCING THE MULTI-QUBIT DYNAMICS INTO A TWO-QUBIT ONE

In this section, we give the definitions of two spin squeezing parameters. By discussing the symmetry of the open system under consideration, we know that the spin squeezing can be expressed by the local expectations and correlations. Since we can reduce hte multi-qubit dynamics into a two-qubit one, the spin squeezing can then calculated by the dynamics of the local expectations and correlations.

A. Spin squeezing definitions

There are various measures of spin squeezing related to various inequality criteria [1–3, 5, 8], and we consider two of them as follows:

$$\xi_{\mathrm{KU}}^2 = \frac{4(\Delta J_\perp)^2_{\min}}{N}, \tag{6}$$

$$\xi_{\mathrm{T}}^2 = \frac{\lambda_{\min}}{\langle \vec{J}^2\rangle - \frac{N}{2}}. \tag{7}$$

Here, the minimization in the first equation is over all the directions denoted by $\perp$, which are perpendicular to the mean spin direction $\langle \vec{J}\rangle/|\langle \vec{J}\rangle|$. $\lambda_{\min}$ in the second equation is the minimal eigenvalue of the matrix

$$\Gamma = (N-1)\gamma + \mathbf{C}, \tag{8}$$

where

$$\gamma_{kl} = \mathbf{C}_{kl} - \langle J_k\rangle\langle J_l\rangle, \quad k, l \in \{x, y, z\} \tag{9}$$

is the covariance matrix and

$$\mathbf{C}_{kl} = \frac{1}{2}\langle J_l J_k + J_k J_l\rangle \tag{10}$$

is the global correlation matrix. The parameters ξ_{KU}^2 and ξ_{T}^2 were defined by Kitagawa and Ueda [1], and Tóth et al. [5], respectively. If $\xi_{\mathrm{T}}^2 < 1$, spin squeezing occurs, and we can safely say that the multipartite state is entangled [5, 8].

From the definitions, we know that the spin squeezing parameters are based on the expectations and correlations of the collective operators. For the limitation of the hierarchy equation method, it is hard to calculate the decoherence of the many-particle system straightforwardly.

B. Simplification of the spin squeezing parameters

Since the baths are independent and identical, the exchange symmetry is not affected by decoherence. Therefore, the global expectations or correlations of collective operators can be written as [39]

$$\langle J_\alpha\rangle = \frac{N}{2}\langle\sigma_{1\alpha}\rangle, \tag{11}$$

$$\langle J_\alpha^2\rangle = \frac{N}{4} + \frac{N(N-1)}{4}\langle\sigma_{1\alpha}\sigma_{2\alpha}\rangle, \tag{12}$$

$$\langle [J_\alpha, J_\beta]_+\rangle = \frac{N(N-1)}{4}\langle[\sigma_{1\alpha}, \sigma_{2\beta}]_+\rangle, \ (\alpha \neq \beta), \tag{13}$$

which only depend on the expectation values of the local Pauli operators, e.g., $\langle\sigma_{1\alpha}\sigma_{2\beta}\rangle$ and $\langle\sigma_{1\alpha}\rangle$.

The initial one-axis twisted state we use here has a parity symmetry leading to $\langle J_x\rangle = \langle J_y\rangle = 0$, namely the mean-spin direction is along the z-axis. Moreover, the mean-spin direction do not change during decoherence. The proof is given as follows.

The Hamiltonian (1) displays only one symmetry, i.e., the parity symmetry. The parity operator is given by

$$\begin{aligned}\Pi &= \Pi_1 \otimes \Pi_2 \\ &= (-1)^{\mathcal{N}} \otimes (-1)^{\sum_k a_k^\dagger a_k} \\ &= (-1)^{\mathcal{N}+\sum_k a_k^\dagger a_k},\end{aligned} \tag{14}$$

where $\mathcal{N} = J_z + N/2$ describes the numbers of excitations of up spins. Obviously, we have

$$\Pi H \Pi = H, \tag{15}$$
$$\Pi_1 \rho_S(0) \Pi_1 = \rho_S(0), \tag{16}$$
$$\Pi_2 \rho_B(0) \Pi_2 = \rho_B(0), \tag{17}$$
$$\Pi \rho_T(0) \Pi = \rho_T(0), \tag{18}$$

namely, the Hamiltonian and the initial state have a fixed parity. Since the exchange symmetry leads to $\langle J_x\rangle = N\langle\sigma_{1x}\rangle/2$, we obtain

$$\begin{aligned}\langle\sigma_{1x}\rangle &= \mathrm{Tr}\left[\sigma_{1x} U(t)\rho_T(0)U^\dagger(t)\right] \\ &= \mathrm{Tr}\left\{\sigma_{1x}\Pi\left[\Pi U(t)\Pi\right]\left[\Pi\rho_T(0)\Pi\right]\left[\Pi U^\dagger(t)\Pi\right]\Pi\right\} \\ &= \mathrm{Tr}\left[\sigma_{1x}\Pi U(t)\rho_T(0)U^\dagger(t)\Pi\right] \\ &= \mathrm{Tr}\left[\Pi\sigma_{1x}\Pi U(t)\rho_T(0)U^\dagger(t)\right] \\ &= -\langle\sigma_{1x}\rangle,\end{aligned} \tag{19}$$

which leads to $\langle J_x\rangle = 0$. Similarly, $\langle J_y\rangle = \langle J_y J_z\rangle = \langle J_x J_z\rangle = 0$ can be proved. Therefore, during the evolution the mean spin direction is always along the z-axis. In this case, the spin squeezing parameters reduce to [7, 28]

$$\xi^2_{\mathrm{KU}} = 1 + 2(N-1)(\langle\sigma_{1+}\sigma_{2-}\rangle - |\langle\sigma_{1-}\sigma_{2-}\rangle|), \tag{20}$$
$$\xi^2_{\mathrm{T}} = \frac{\min\left\{\xi^2_{KU}, \varsigma^2\right\}}{(1-1/N)\langle\vec{\sigma}_1\cdot\vec{\sigma}_2\rangle + 1/N}, \tag{21}$$

where

$$\varsigma^2 = 1 + (N-1)\left(\langle\sigma_{1z}\sigma_{2z}\rangle - \langle\sigma_{1z}\rangle\langle\sigma_{2z}\rangle\right). \tag{22}$$

For convenience, hereafter we use

$$\zeta_k^2 = \max(0, 1-\xi_k^2), \quad k \in \{\mathrm{KU}, \mathrm{T}\} \tag{23}$$

to characterize spin squeezing. With the above definition, spin squeezing occurs when $\zeta_k^2 > 0$.

Now we only need to calculate the dynamics of the local expectations and correlations of the spins, the spin squeezing parameters are greatly simplified. However, we still have to use the dynamics of the density matrix of the system to calculate the local expectations and correlations.

C. Reducing the multi-qubit dynamics into a two-qubit one

Now we prove that we can reduce the multi-qubit dynamics into a two-qubit one. Generally, we consider a system written as follows

$$H = \sum_{i=1}^N H^{(i)},\ H^{(i)} = H_S^{(i)} + H_B^{(i)} + H_{SB}^{(i)}. \tag{24}$$

$H_S^{(i)}$ is the Hamiltonian of one particle, $H_B^{(i)}$ is the bath Hamiltonian, and the couplings are expressed by $H_{SB}^{(i)}$. Obviously, each of the particles interacts with its own bath. The particles do not have interaction with each other, and the baths are independent. Equation (1) belongs to this case.

The time-evolution operator of the total system can be written as

$$U(t) = e^{-iHt} = \prod_i e^{-iH_i t} = \prod_i u_i(t), \tag{25}$$

where $u_i(t) = e^{-iH_i t}$. Then, the total density matrix at time t is given by

$$\rho_T(t) = U(t)\rho_{\mathrm{T}}(0)U^\dagger(t), \tag{26}$$

which can be formally written as

$$\begin{aligned}\rho_T(t) &= U(t)\rho_T(0)U^\dagger(t) \\ &= \prod_i u_i(t)\rho_T(0)\prod_i u_i^\dagger(t).\end{aligned} \tag{27}$$

Here we assume that the initial state is a product state written as

$$\rho_T(0) = \rho_S(0) \otimes \rho_B(0). \tag{28}$$

By tracing out the baths and $N-2$ particles of the system, we obtain the reduced density matrix of any two particles

$$\begin{aligned}&\rho_S^{12}(t) \\ &= \mathrm{Tr}_{\{B_{1,2}\}}\left[\mathrm{Tr}_{\{S_{3\ldots N}B_{3\ldots N}\}}\left(\prod_{i=1}^N u_i(t)\rho_T(0)\prod_{i=1}^N u_i^\dagger(t)\right)\right] \\ &= \mathrm{Tr}_{\{B_{1,2}\}}\left[\mathrm{Tr}_{\{S_{3\ldots N}B_{3\ldots N}\}}\left(\prod_{i=1}^2 u_i(t)\rho_T(0)\prod_{i=1}^2 u_i^\dagger(t)\right)\right] \\ &= \mathrm{Tr}_{\{B_{1,2}\}}\left[\prod_{i=1}^2 u_i(t)\left(\rho_S^{12}(0)\otimes\rho_B^{12}(0)\right)\prod_{i=1}^2 u_i^\dagger(t)\right],\end{aligned} \tag{29}$$

where the second equality follows from the fact (see Appendix A)

$$\begin{aligned}&\mathrm{Tr}_2\left[(A_1\otimes A_2)\,\rho_{12}\,(B_1\otimes B_2)\right] \\ &= \mathrm{Tr}_2\left[A_1\otimes(B_2A_2)\rho_{12}\,(B_1\otimes I_2)\right],\end{aligned} \tag{30}$$

and the last equality is obtained by substituting the initial product state (28). $\rho_S^{12}(0) = \mathrm{Tr}_{\{S_{3\ldots N}\}}\rho_S(0)$ and $\rho_B^{12}(0) = \mathrm{Tr}_{\{B_{3\ldots N}\}}\rho_B(0)$ in the equation are the reduced density matrices of the initial state for the system and baths respectively.

The above equation (30) tells that the evolution of any two particles is governed only by the local Hamiltonian of the two particles and their baths. It is noted that we can reach this conclusion even when the initial state of the system or the baths are entangled states. Therefore, the multi-qubit dynamics reduces to the two-qubit one. Then we use the hierarchy equation method to calculate the two-qubit reduced density matrix of the system, and the dynamics of the local expectations and correlations in Eqs. (20)-(22) can also be obtained.

Here we emphasize that we obtain this conclusion without using exchange symmetry, which means that the particles are not necessarily identical, and so do the baths. Also, the proof can be easily extended to any finite number of particles.

IV. HIERARCHY EQUATIONS AND INITIAL TWO-QUBIT REDUCED DENSITY MATRIX

To start with the numerical calculations, we introduce the hierarchy equation method [36, 37] and discuss the spin squeezing parameters of the initials state in this section. For comparison, the definition of a rescaled concurrence is also given.

A. Hierarchy equations

We choose the Drude-Lorentz spectrum,

$$J(\omega) = \frac{2}{\pi}\frac{\omega\lambda\gamma}{\omega^2+\gamma^2}, \tag{31}$$

where γ represents the width of the spectral distribution of the bath mode and λ can be viewed as the system-bath coupling strength. The bath correlation function for the bath operator

$$B_\alpha(t) = \sum_k g_{\alpha k}\left(b_k^\dagger e^{i\omega_k t} + b_k e^{-i\omega_k t}\right) \tag{32}$$

is given by [37]

$$\langle B_\alpha(t) B_\alpha(\tau)\rangle = \sum_{n=0}^{\infty} c_n e^{-\nu_n |t-\tau|}, \tag{33}$$

where

$$\nu_k = \frac{2\pi k}{\beta}(1-\delta_{k0}) + \gamma\delta_{k0}, \tag{34}$$

is the k-th Matsubara frequency, and

$$c_k = \frac{4\lambda\gamma}{\beta}\frac{\nu_k}{\nu_k^2-\gamma^2}(1-\delta_{k0}) + \lambda\gamma\left[\cot\left(\frac{\beta\gamma}{2}\right) - i\right]\delta_{k0} \tag{35}$$

are the expansion coefficients.

With the Drude-Lorentz spectrum, the hierarchy equations becomes

$$\begin{aligned}\dot{\rho}_{\vec{n}} &= -\left[iH_S^\times + (\vec{n}_1+\vec{n}_2)\cdot\vec{\nu}\right]\rho_{\vec{n}} \\ &\quad -\left(\frac{2\lambda}{\beta\gamma} - i\lambda - \sum_{k=0}^{M}\frac{c_k}{\nu_k}\right)V_\alpha^\times V_\alpha^\times \rho_{\vec{n}} \\ &\quad -i\sum_{\alpha=1}^{2}\sum_{k=0}^{M} n_{\alpha k}\left(c_k V_\alpha \rho_{\vec{n}-\vec{e}_{\alpha k}} - c_k^* \rho_{\vec{n}-\vec{e}_{\alpha k}} V_\alpha\right) \\ &\quad -i\sum_{\alpha=1}^{2}\sum_{k=0}^{M} V_\alpha^\times \rho_{\vec{n}+\vec{e}_{\alpha k}},\end{aligned} \tag{36}$$

where

$$\vec{n} = (\vec{n}_1, \vec{n}_2) = (n_{10}, ..., n_{1M}, n_{20}, ..., n_{2M}) \tag{37}$$

is a $2(M+1)$-dimensional vector, a concatenation of two $(M+1)$-dimensional vectors $\vec{n}_1$ and $\vec{n}_2$. The vectors $\vec{\nu} = (\nu_0, ...\nu_M)$ and $\vec{e}_{\alpha k}$ are defined as $2(M+1)$-dimensional vectors with only 1 in the αk place and 0s in other places. Note that this equation is slightly different and essentially the same as that given in Ref. [37].

B. Initial two-qubit reduced density matrix

To solve Eq. (36), we need the initial state. Since the mean spin of the initial state (4) is along the z-direction, the two-qubit reduced density matrix can be written as a block-diagonal form [7],

$$\rho_{12} = \begin{pmatrix} v_+ & u^* \\ u & v_- \end{pmatrix} \oplus \begin{pmatrix} w & y \\ y & w \end{pmatrix}, \tag{38}$$

in the basis $\{|00\rangle, |11\rangle, |01\rangle, |10\rangle\}$, where

$$v_\pm = \left(1 \pm 2\langle\sigma_{1z}\rangle + \langle\sigma_{1z}\sigma_{2z}\rangle\right)/4, \tag{39}$$
$$w = \left(1 - \langle\sigma_{1z}\sigma_{2z}\rangle\right)/4, \tag{40}$$
$$u = \langle\sigma_{1-}\sigma_{2-}\rangle, \tag{41}$$
$$y = \langle\sigma_{1+}\sigma_{2-}\rangle. \tag{42}$$

We notice that if $\langle\sigma_{1+}\sigma_{2-}\rangle$, $\langle\sigma_{1-}\sigma_{2-}\rangle$, $\langle\sigma_{1z}\rangle$, and $\langle\sigma_{1z}\sigma_{2z}\rangle$ are known, the density matrix is determined. For the one-axis twisted state, we have [7]

$$\langle\sigma_z\rangle = -\cos^{N-1}\left(\frac{\theta}{2}\right), \tag{43}$$
$$\langle\sigma_{1z}\sigma_{2z}\rangle = \frac{1}{2}\left(1+\cos^{N-2}\theta\right), \tag{44}$$
$$\langle\sigma_{1+}\sigma_{2-}\rangle = \frac{1}{8}\left(1-\cos^{N-2}\theta\right), \tag{45}$$
$$\begin{aligned}\langle\sigma_{1-}\sigma_{2-}\rangle &= -\frac{1}{8}\left(1-\cos^{N-2}\theta\right) \\ &\quad -\frac{i}{2}\sin\left(\frac{\theta}{2}\right)\cos^{N-2}\left(\frac{\theta}{2}\right).\end{aligned} \tag{46}$$

Employing the equations above, we obtain the initial two-qubit reduced density matrix in Eq. (38). Then we use Eq. (36) to calculate the dynamics of the reduced density matrix numerically.

Meanwhile, we can also use the Eqs. (43)-(46) to discuss the spin squeezing parameters for the initial state. For the initial state (4), we obtain

$$\zeta_{\mathrm{KU}}^2(0) = \zeta_{\mathrm{T}}^2(0) = \frac{1}{4}\left\{\left[(1-\cos^{N-2}\theta)^2 + 16\sin^2\left(\frac{\theta}{2}\right) \times \cos^{2N-4}\left(\frac{\theta}{2}\right)\right]^{1/2} - 1 + \cos^{N-2}\theta\right\}, \quad (47)$$

which implies that the two spin squeezing parameters for the initial state coincide.

It is known that the spin squeezing has close relation with concurrence if the state of the collective spin system lies in the $J = N/2$ sector [39], such as the initial state of the system. During the decoherence, the state of the system does not lie in $J = N/2$ sector anymore. It is necessary to compare the behaviors of spin squeezing and pairwise entanglement.

The concurrence is defined as [40]

$$C = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4), \quad (48)$$

where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ are the square roots of eigenvalues of $\tilde{\rho}\rho$. Here ρ is the reduced density matrix of the system, and

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y), \quad (49)$$

where ρ^* is the conjugate of ρ.

For the reduced density matrix of (38), the concurrence is given by [41]

$$C = 2\max\left\{0, |u| - y, y - \sqrt{v_+ v_-}\right\}. \quad (50)$$

Therefore, we can also obtain the concurrence of the initial state by employing Eqs. (39)-(46).

For convenience, here we use a rescaled concurrence

$$C_r = (N-1)C, \quad (51)$$

and thus $C_r(0) = \zeta_{\mathrm{KU}}^2(0) = \zeta_{\mathrm{T}}^2(0)$. Then we know that the two spin squeezing parameters and concurrence are same for the initial state.

V. SPIN SQUEEZING AND CONCURRENCE UNDER DECOHERENCE

The initial one-axis twisted state considered in this work is a symmetric state which can be expressed as a superposition of symmetric Dicke states. In other words, the N qubits behave effectively like a large spin $N/2$. After decoherence, not only the symmetric Dicke states will be populated, but also states with lower symmetry. Therefore, it is not sufficient to describe the system with only an $(N+1)$-dimensional space. However, the exchange symmetry is not affected by the decoherence. In other words, a state with exchange symmetry does not necessarily belong to the maximally-symmetric space [42]. Now by employing the hierarchy equation method, we calculate the spin squeezing parameters and the rescaled concurrence under decoherence, and compare the behaviors of them.

As an example, we set the initial state given in Eq. (47)

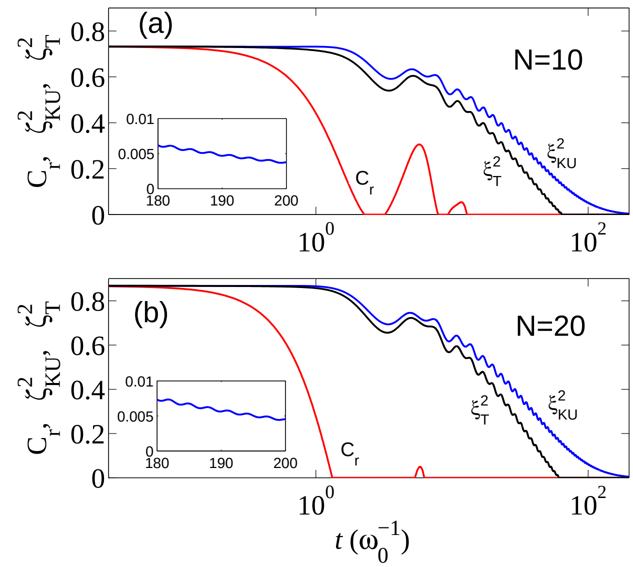

FIG. 1: (Color online) Time evolution of the spin squeezing parameters ζ_{KU}^2, ζ_{T}^2, and the rescaled concurrence C_r for (a) $N = 10$ and (b) $N = 20$. The inverse temperature is taken as $\beta = 4/\omega_0$. The insets in the figures show the magnification of the region where ζ_{KU}^2 nearly vanishes. The horizontal x axes are logarithmic, but the inset x axes are linear.

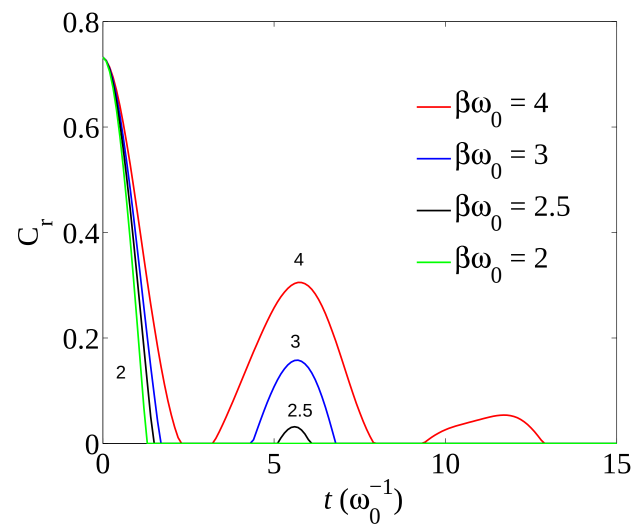

FIG. 2: (Color online) Time evolution of the rescaled concurrence for different values of the inverse temperature β. Here, we choose $N = 10$.

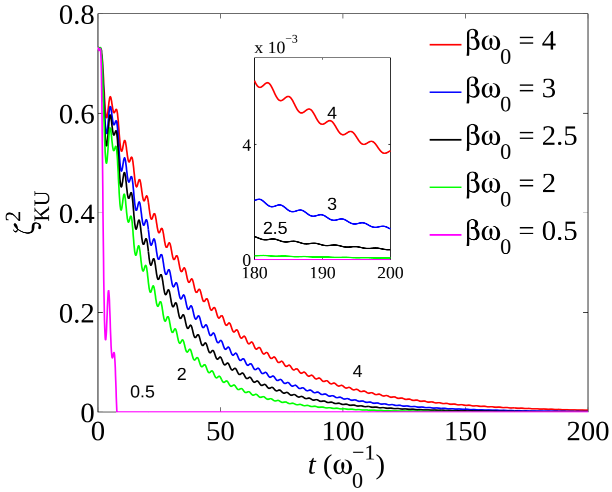

FIG. 3: (Color online) Time evolution of the spin squeezing parameter $\zeta^2_{\rm KU}$ for different values of β, with $N = 10$. The inset shows the magnification of the region when t is large.

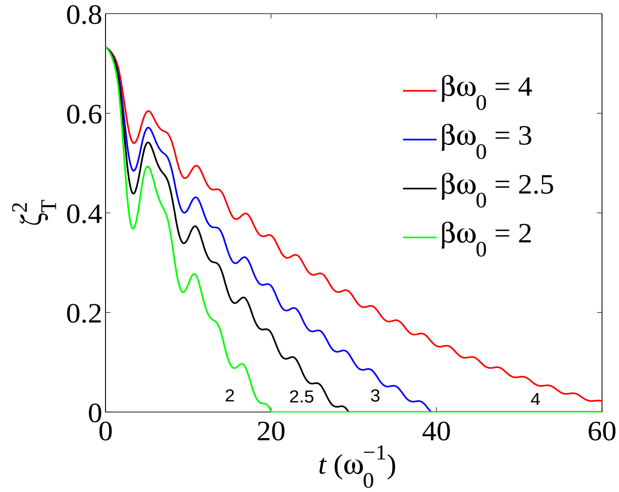

FIG. 4: (Color online) Time evolution of the spin squeezing parameter $\zeta^2_{\rm T}$ for different values of β, with $N = 10$.

with $\theta = \pi/10$. The parameters of the Drude-Lorentz spectrum in Eq. (31) are chosen to be $\lambda = 0.03\omega_0$ and $\gamma = 0.15\omega_0$. In this section, we study the effect of the particle number N and bath temperature T on the dynamics of spin squeezing and concurrence.

Figures 1a and 1b show the time evolution of $\zeta^2_{\rm KU}$, $\zeta^2_{\rm T}$ and C_r with two different particle number $N = 10$ and $N = 20$. The inverse temperature is set to be $\beta = 4/\omega_0$. The figures show that the decay rate of C_r increases with N. Although the rescaled concurrence of the initial state for $N = 20$ is larger than that for $N = 10$, it vanishes earlier. Also, the revival, after a sudden vanishing, becomes weaker with increasing N. Both $\zeta^2_{\rm KU}$ and $\zeta^2_{\rm T}$ decay in an oscillatory way. We observe that $\zeta^2_{\rm T}$ vanishes suddenly, while interestingly, $\zeta^2_{\rm KU}$ decays to zero asymptotically as shown in the insets. Comparing Fig 1a and 1b , we find that for spin squeezing, the vanishing time changes little with increasing N.

Now we focus on the effects of the bath temperature on the dynamics of spin squeezing and rescaled concurrence, which are shown by Figs. 2-4. These figures are plotted with a fixed particle number $N = 10$ and different temperature T. Here we choose the inverse temperature $\beta = 4/\omega_0, 3/\omega_0, 2.5/\omega_0, 2/\omega_0$, and we specially take $\beta = 0.5/\omega_0$ for $\zeta^2_{\rm KU}$. Firstly, let us discuss the time evolutions of C_r which are shown in Fig. 2. As expected, C_r is suppressed with increasing temperature. When we choose a low temperature, such as $\beta = 4/\omega_0$, C_r decays with multiple revivals. When the temperature increases, the revivals become weaker. C_r even vanishes completely without revival when $\beta = 1/\omega_0$.

The spin squeezing is also suppressed with increasing T. As shown in Fig. 3, $\zeta^2_{\rm KU}$ decays without sudden vanishing and approaches zero asymptotically ($t \to \infty$) when temperature is not high enough, which is shown in the inset. Interestingly, when temperature reaches to $\beta = 0.5/\omega_0$, $\zeta^2_{\rm KU}$ decays to zero quickly and suddenly without revival. The behavior is quite different with C_r. While $\zeta^2_{\rm T}$ decays and suddenly vanishes even with low temperature as shown in Fig. 4, which is similar to C_r.

From the comparison, we find that the spin squeezing is not a satisfactory indicator of pairwise entanglement under decoherence for the open systemr, although they have close relations. Also, it is noted that the spin squeezing and concurrence both decay with oscillations, which is a reflection of the non-Markovian dynamics of the system.

VI. CONCLUSION

In this work, we consider an ensemble of N spin-1/2 particles interacting with identical independent bosonic heat baths. The one-axis twisted state is chosen to be the initial state. The mean spin direction of the initial state is along the z-axis, and it does not change during the decoherence dynamics. For the open system we consider, we proved that the multi-qubit dynamics can be reduced into a two-qubit one. Then we use the hierarchy equation method to study the spin squeezing and concurrence under decoherence. This is an exact method without using rotating-wave and the Born-Markov approximation.

From the numerical results, we find that the decay rate of the rescaled concurrence increases with the particle number N as well as the bath temperature T, and the revivals become weaker over time. For the spin squeezing, it is suppressed with increasing temperature as expected, while the vanishing time changes little with N. The spin squeezing parameter $\zeta^2_{\rm KU}$ vanishes asymptotically with low bath temperature and disappear suddenly when bath temperature is high enough. Interestingly, $\zeta^2_{\rm T}$ vanishes suddenly even when bath temperature is low, which is similar to C_r.

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Appendix A: A formula on the partial trace

It should be noted that the second term of Eq. (29) is obtained by moving $\prod_{i=3}^{N} u_i^{\dagger}(t)$ from the right of $\rho_{tot}(0)$ to the left, which use the relation of Eq. (30). We now prove this property of partial trace. Considering a two-subspace case as an example:

$$\begin{aligned}
& \mathrm{Tr}_2\left[\left(A_1 \otimes A_2\right) \rho_{12} \left(B_1 \otimes B_2\right)\right] \\
= & \sum_{n} \left[\left(A_1 \otimes \left(\langle n| A_2\right) \rho_{12} \left(B_1 \otimes \left(B_2 |n\rangle\right)\right)\right)\right] \\
= & \sum_{nm} \left[A_1 \otimes \left(\langle n| A_2 \rho_{12} B_1\right) \otimes \left(|m\rangle \langle m| B_2 |n\rangle\right)\right] \\
= & \sum_{nm} \left[\left(A_1 \otimes \left(\langle m| B_2 |n\rangle \langle n| A_2\right) \rho_{12} B_1 \otimes |m\rangle\right]\right. \\
= & \sum_{m} \left[A_1 \otimes \left(\langle m| B_2 A_2\right) \rho_{12} B_1 \otimes |m\rangle\right] \\
= & \mathrm{Tr}_2\left[A_1 \otimes \left(B_2 A_2 \rho_{12} \left(B_1 \otimes I_2\right)\right]\right., \qquad \text{(A1)}
\end{aligned}$$

where $I_{1,2}$ is the identity matrix of the 1 or 2 subspace.

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